Applied Mathematics and Statistics

Foundation Qualifying Examination Part B

in Computational Applied Mathematics

Spring 2018 (January)

(Closed Book Exam)

Please solve 3 out of 4 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part B: 1 2 3 4

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: January 22, 2018

Time: 11:00 AM – 12:00 PM

B1. (10 points) Find n independent asymptotic solutions (leading behaviors) to the differential equation

$$\frac{d^n y}{dx^n} = \frac{y}{x^{2n}}, \quad \text{as } x \to 0.$$

B2. (10 points) The Lagrangian for a system of two particles with masses m_1 and m_2 and coordinates $\mathbf{r}_1$ and $\mathbf{r}_1$, interacting via a potential $V(\mathbf{r}_1 - \mathbf{r}_2)$, is

$$L = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - V(\mathbf{r}_1 - \mathbf{r}_2).$$

a) Rewrite the Lagrangian in terms of the center of mass coordinates $\mathbf{R}$ and the relative coordinates $\mathbf{r}$ defined as

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

b) Derive Lagrange's equations of motion in terms of $\mathbf{R}$ and $\mathbf{r}$.

c) Find solutions to Lagrange's equations (if an equation depends on unknown $V(\mathbf{r})$, express its solution in integral form), and explain your results.

B3. Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, and $B \in \mathbb{R}^{n \times n}$ is symmetric.

a) (4 points) Show that the matrix AB is diagonalizable, and all its eigenvalues are real.

b) (3 points) Show that the matrices AB and BA have the same set of eigenvalues.

c) (3 points) Are the eigenvectors of AB orthogonal to each other? Why or why not?

B4. (10 points) Let $A \in \mathbb{C}^{m \times m}$ be Hermitian positive definite matrix with Cholesky factorization $A = R^* R$. Show that $\sqrt{\kappa_2(A)} = \kappa_2(R)$, where $\kappa_2(X)$ denotes the condition number of matrix X measured in 2-norm.